Refinement of Two-Factor Factorizations of a Linear Partial Differential Operator of Arbitrary Order and Dimension

Ekaterina Shemyakova

Abstract. Given a right factor and a left factor of a Linear Partial Differential Operator (LPDO), under which conditions we can refine these two-factor factorizations into one three-factor factorization? This problem is solved for LPDOs of arbitrary order and number of variables. A more general result for the incomplete factorizations of LPDOs is proved as well.

Mathematics Subject Classification (2000). Primary 47F05; Secondary 68W30.

Keywords. factorization, LPDOs, PDEs, Linear Partial Differential Operators.

1. Introduction

The factorization of Linear Partial Differential Operators (LPDOs) is an essential part of recent algorithms for the exact solution for Linear Partial Differential Equations (LPDEs). Examples of such algorithms include numerous generalizations and modifications of the 18th-century Laplace Transformations Method [24, 23, 21, 1, 2, 3, 4, 20], the Loewy decomposition method [11, 12, 13], and others.

The problem of constructing a general factorization algorithm for an LPDO is still an open problem, although several important contributions have been made in recent decades, and different approaches have been applied (see [11, 15, 14, 22, 23, 19, 6, 8, 7] and many others). Many of the recent approaches are concerned, in particular, with explaining the non-uniqueness of factorization: (irreducible) factors and the number of factors are not necessarily the same for two different factorizations of the same operator. This is commonly illustrated by the famous example of Landau [5],

Example (Landau).

$$\begin{aligned} L &= \left(D_x + 1 + \frac{1}{x+c(y)}\right) \circ \left(D_x + 1 - \frac{1}{x+c(y)}\right) \circ (D_x + xD_y) = \\ &= (D_{xx} + xD_{xy} + D_x + (2+x)D_y) \circ (D_x + 1)\ , \end{aligned}$$

where the second-order factor in the second factorization is hyperbolic and is irreducible.

On the other hand, for some classes of LPDOs factorization is unique. For example, there is no more than one factorization that extends a factorization of the principal symbol of the operator into co-prime factors [11]. Algebraic theories have been introduced to explain this phenomenon theoretically; see Tsarev [21], Grigoriev and Schwarz [13] and most recently Cassidy and Singer [16].

Some important methods of exact integration, for example, the Loewy decomposition methods mentioned above, require LPDOs to have a number of different factorizations of certain types. Also completely reducible LPDOs introduced in [11], which become significant as the solution space of a completely reducible LPDO coincides with the sum of those of its irreducible right factors may require a number of right factors.

In earlier work [17] we have exhaustively studied families of factorizations for operators up to order 4, and described when the same operator has multiple factorizations of *the same factorization type*, to be more specific, when there exist an infinite number of factorizations of the same factorization type, meaning having the same symbols of the factors. The first non-trivial example of such families of order 4 has been found:

Example. [17] The following is a fourth-order irreducible family of factorizations:

$$D_{xxyy} = \left(D_x + \frac{\alpha}{y+\alpha x+\beta}\right)\left(D_y + \frac{1}{y+\alpha x+\beta}\right)\left(D_{xy} - \frac{1}{y+\alpha x+\beta}(D_x + \alpha D_y)\right),$$

where $\alpha, \beta \neq 0$. Note that the first two factors commute. So the operator D_{xxyy} has a family of factorizations, and every factorization of the family is of the same factorization type $(X)(Y)(XY)$, that is the highest order terms in the first, the second and the third factors are D_x, D_y and D_{xy} correspondingly.

In recent work [10] non-uniqueness of a different kind is addressed. There, we considered factorizations of *different factorization types*, and by using invariants proved that a third-order bivariate operator L has a first-order left factor of the symbol S_1 and a first-order right factor of the symbol S_2, where $\gcd(S_1, S_2) = 1$ if and only if it has a complete factorization of the type $(S_1)(T)(S_2)$, where $T = \mathrm{Sym}(L)/(S_1 S_2)$. Further investigations in the same paper show that a third-order bivariate operator L has a first-order left factor F_1 and a first-order right factor F_2 with $\gcd(\mathrm{Sym}(F_1), \mathrm{Sym}(F_2)) = 1$ if and only if L has a factorization into three factors, the left one of which is exactly F_1 and the right one is exactly F_2.

Example. [10] The existence of two factorizations for an LPDO,

$$(D_x + x) \circ (D_{xy} + yD_x + y^2 D_y + y^3) = A = (D_{xx} + (x+y^2)D_x + xy^2) \circ (D_y + y)$$

implies the existence of the "complete" factorization of A,

$$A = (D_x + x) \circ (D_x + y^2) \circ (D_y + y) \ .$$

On the other hand, if the condition $\gcd(\mathrm{Sym}(F_1), \mathrm{Sym}(F_2)) = 1$ fails, then it can happen that the "complete" factorization does not exist.
Example. [10]

$$(D_x D_y + 1) \circ (D_x + 1) = (D_x + 1) \circ (D_x D_y + 1) \ ,$$

while $D_x D_y + 1$ has no factorization at all.

In the present paper we have generalized the result of [10] to the case of LPDOs of arbitrary order and of arbitrary dimension. Moreover, a more general statement has been formulated and proved for incomplete factorizations of LPDOs. We describe the results in terms of common obstacles, which we have introduced in [19].

2. Preliminaries

Consider a field K of characteristic zero with commuting derivations $\partial_1, \dots, \partial_n$, and the corresponding non-commutative ring of linear partial differential operators (LPDOs) $K[D] = K[D_1, \dots, D_n]$, where D_i corresponds to the derivation ∂_i for all $i \in \{1, \dots, n\}$. In $K[D]$ the variables $D_1, \dots, D_n$ commute with each other, but not with elements of K. We write multiplication in $K[D]$ as "$\circ$"; i.e. $L_1 \circ L_2$ for $L_1, L_2 \in K[D]$. Any operator $L \neq 0 \in K[D]$ has the form

$$L = \sum_{|J|=0}^{d} a_J D^J \ , a_J \in K \ , \tag{2.1}$$

where $J = (j_1, \dots, j_n)$ is a multi-index in $\mathbb{N}^n$, $|J| = j_1 + \dots + j_n$, and where $D^J = D_1^{j_1} \dots D_n^{j_n}$. Further, there exists some J, with $|J| = d$, such that $a_J \neq 0$. Then d is the order of L. For the case $L = 0$ we define the order as $-\infty$.

When considering the bivariate case $n = 2$, we use the following formal notations: $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\partial_1(f) = f_x$, $\partial_2(f) = f_y$, where $f \in K$, and correspondingly $D_1 \equiv D_x$, and $D_2 \equiv D_y$ for ease of notation.

For an operator $L \neq 0$ of the form (2.1) the homogeneous commutative polynomial

$$\mathrm{Sym}(L) = \sum_{|J|=d} a_J X^J \tag{2.2}$$

in formal variables $X_1, \dots, X_n$ is called the (*principal*) *symbol*, and if $L = 0$, the symbol is defined to be zero. Vice versa, given a homogeneous commutative polynomial $S \in K[X]$ in the form (2.2), we define the operator $\widehat{S} \in K[D]$ as the result of substituting D_i for each variable X_i.

3. Main Result

Since for two LPDOs $L_1, L_2 \in K[D]$ we have $\mathrm{Sym}(L_1 \circ L_2) = \mathrm{Sym}(L_1) \cdot \mathrm{Sym}(L_2)$, any factorization of an LPDO extends some factorization of its symbol. In general, if $L \in K[D]$ and $\mathrm{Sym}(L) = S_1 \dots S_k$, let us say that the factorization

$$L = F_1 \circ \dots \circ F_k \ , \quad \mathrm{Sym}(F_i) = S_i, \ \forall i \in \{1, \dots, k\} \ ,$$

is of the factorization type $(S_1) \dots (S_k)$.

For the second-order hyperbolic LPDOs, which have normalized form

$$L = D_x D_y + a D_x + b D_y + c \ , \tag{3.1}$$

where $a, b, c \in K$, it is common to consider their incomplete factorizations:

$$L = (D_x + b) \circ (D_y + a) + h = (D_y + a) \circ (D_x + b) + k \ ,$$

where $h = c - a_x - ab$ and $k = c - b_y - ab$ are invariants of (3.1) with respect to gauge transformations, $L \to g^{-1} L g$, $g \neq 0, g \in K$ and are called the Laplace invariants. This is an element in the foundation of the classical Laplace-Darboux-Transformations Method [9].

In [18, 19] a generalization of this idea is suggested. Thus, for $A \in K[D]$ with $\mathrm{Sym}(A) = S_1 \dots S_k$, we call [18, 19] an LPDO $R \in K[D]$ a *common obstacle* to factorization of the type $(S_1)(S_2) \dots (S_k)$ if there exists a factorization of this type for the operator $A - R$, and R has minimal possible order.

The following example demonstrates different possibilities for common obstacles and incomplete factorizations.

Example. Consider the LPDO

$$A_4 = D_x^2 D_y^2 + D_x + D_y + 1 \ . \tag{3.2}$$

1. Unique common obstacle and unique incomplete factorization. Consider factorizations of A_4 of the factorization type $(X^2)(Y^2)$. Assume that the order of common obstacles is one or less (if we come to a contradiction, we have to search then for higher-order common obstacles), and search for common obstacles in the form $R_1 = p_1 D_x + q_1 D_y + r_1$, where $p_1, q_1, r_1 \in K$. Thus, for some $l_{10}, l_{01}, l_{00}, f_{10}, f_{01}, f_{00} \in K$ we have

$$A_4 = (D_x^2 + l_{10} D_x + l_{01} D_y + l_{00}) \circ (D_y^2 + f_{10} D_x + f_{01} D_y + f_{00}) + R_1 \ .$$

Comparing the corresponding coefficients we have $l_{10} = l_{01} = l_{00} = f_{10} = f_{01} = f_{00} = 0$, $p_1 = q_1 = r_1 = 1$, that is, there is a unique common obstacle and a unique incomplete factorization of the factorization type $(X^2)(Y^2)$,

$$A_4 = D_x^2 \circ D_y^2 + D_x + D_y + 1 \ .$$

2. Infinitely many common obstacles and incomplete factorizations. Consider factorizations of A_4 of the factorization type $(X)(XY^2)$. Again assume that the order of common obstacles is one or less, and search for common obstacles in the form $R_2 = p_2 D_x + q_2 D_y + r_2$, where $p_2, q_2, r_2 \in K$. Thus, for some $m_{00}, g_{ij} \in K$ we have $A_4 = (D_x + m_{00}) \circ (D_x D_y^2 + \sum_{i+j=0}^{2} g_{ij} D_x^i D_y^j) + R_2$. Comparing the

corresponding coefficients we have $g_{02} = -m_{00}$, $g_{20} = g_{11} = g_{10} = g_{01} = 0$, $q_2 = 1$, $p_2 = 1 - g_{00}$, $r_2 = 1 - m_{00}g_{00} - g_{00x}$, while m_{00} satisfies $m_{00}^2 + m_{00x} = 0$, and g_{00} is a free parameter. Thus, we have

$$A_4 = (D_x + m_{00}) \circ (D_x D_y^2 - m_{00} D_y^2 + g_{00}) + (1 - g_{00}) D_x + D_y + 1 - m_{00} g_{00} - g_{00x} \ ,$$

and the order of common obstacles is 1.

3. Unique common obstacle and infinitely many incomplete factorizations. Consider factorizations of A_4 of the factorization type $(X)(X)(Y^2)$. We search for common obstacles in the form $R_3 = p_3 D_x + q_3 D_y + r_3$, where $p_3, q_3, r_3 \in K$. Thus, for some $m_3, n_3, a_3, b_3, c_3 \in K$ we have

$$A_4 = (D_x + m_3) \circ (D_x + n_3) \circ (D_y^2 + a_3 D_x + b_3 D_y + c_3) + R_3 \ .$$

Equating the corresponding coefficients we have $n_3 = -m_3$, $a_3 = b_3 = c_3 = 0$, $p_3 = q_3 = r_3 = 1$, and m_3 satisfies $m_3^2 + m_{3x} = 0$, that is we have a unique common obstacle, but incomplete factorizations can be different:

$$A_4 = (D_x + m_3) \circ (D_x - m_3) \circ D_y^2 + D_x + D_y + 1 \ .$$

The following lemma is used for the proof of Theorem 3.2.

Lemma 3.1 (Division lemma). *Let $L, M \in K[D]$ and $\mathrm{Sym}(L)$ is divisible by $\mathrm{Sym}(M)$, then there exist $N, R \in K[D]$ such that*

$$L = M \circ N + R \ ,$$

where either $R = 0$, or $\mathrm{Sym}(R)$ is not divisible by $\mathrm{Sym}(M)$. Here R is the remainder of the incomplete factorization.

Proof. Let $\mathrm{Sym}(L) = S_1 S_2$, $\mathrm{Sym}(M) = S_1$. Construct a finite sequence of n (for some n) incomplete factorizations of L of the form $L = M \circ N_i + Q_i$, where $\mathrm{Sym}(N_i) = S_2, 1 \le i \le n$, and $\mathrm{Sym}(Q_n)$ is either zero or not divisible by $\mathrm{Sym}(M)$. Start with $N_1 = \widehat{S_2}$ and let $Q_1 = L - M \circ N_1$. If $\mathrm{Sym}(Q_1)$ is either zero or not divisible by $\mathrm{Sym}(M)$, we stop and let $N = N_1$ and $R = Q_1$. Otherwise, let $T_1 = \mathrm{Sym}(Q_1)/\mathrm{Sym}(M)$ and let $N_2 = N_1 + \widehat{T_1}$, and let $Q_2 = Q_1 - M \circ \widehat{T_1}$ (which implies $Q_2 = L - M \circ N_2$). If $\mathrm{Sym}(Q_2)$ is either zero or not divisible by $\mathrm{Sym}(M)$, we stop and let $N = N_2$, $R = Q_2$. Otherwise, we continue in the same manner. Since we clearly have $\mathrm{ord}(Q_2) < \mathrm{ord}(Q_1) < \mathrm{ord}(L)$, and in general, $\mathrm{ord}(Q_{i+1}) < \mathrm{ord}(Q_i)$, this process must stop after a finite, say n, number of steps, and we have $L = M \circ N + R$, where $\mathrm{Sym}(N) = S_2$ and $\mathrm{Sym}(R)$ is either zero or is not divisible by $\mathrm{Sym}(M)$. $\square$

Let $A \in K[D]$ and $\mathrm{Sym}(A) = S_1 \cdot S_2 \cdot S_3$. It is easy to see that every common obstacle to factorization of the type $(S_1)(S_2)(S_3)$ is the remainder for some incomplete factorization of the type $(S_1 S_2)(S_3)$ and so it is for some incomplete factorization of the type $(S_1)(S_2 S_3)$ (the order of the common obstacles for a factorization of the type $(S_1 S_2)(S_3)$ (resp. $(S_1)(S_2 S_3)$) can be smaller that of the

common obstacles for a factorization of the type $(S_1)(S_2)(S_3)$. In general, the inverse statement is not true for it is more difficult to find a factorization into more factors.

The following theorem states that under some conditions, common obstacles to factorization into two factors are the same as those into three factors.

Theorem 3.2. *Let* $A \in K[D]$ *and suppose* $\mathrm{Sym}(A) = S_1 S_2 S_3$*, with* $\gcd(S_1, S_3) = 1$*. Let* U*,* V *and* W *be respectively the sets of common obstacles to factorizations of* A *of the types* $(S_1)(S_2 S_3)$*,* $(S_1 S_2)(S_3)$*, and* $(S_1)(S_2)(S_3)$*. Suppose* V *(resp.* U*) is non-empty and the order of common obstacles in* V *is less than* $\mathrm{ord}(S_3)$*. Then* W *is non-empty and* $V = W$*.*

Proof. Let $R_1 \in V$ be any common obstacle of type $(S_1 S_2)(S_3)$. Then we have $\mathrm{ord}(R_1) < \mathrm{ord}(S_3)$. Let $L, F \in K[D]$ be such that

$$A = L \circ F + R_1 \ , \tag{3.3}$$

where $\mathrm{Sym}(L) = S_1 S_2$, $\mathrm{Sym}(F) = S_3$. Similarly, let $R_2 \in U$, $\mathrm{ord}(R_2) < \mathrm{ord}(S_3)$ be a fixed remainder with respect to an incomplete factorizations of A of type $(S_1)(S_2 S_3)$. Let $M, G \in K[D]$ be such that

$$A = M \circ G + R_2 \ , \tag{3.4}$$

where $\mathrm{Sym}(M) = S_1$, $\mathrm{Sym}(G) = S_2 S_3$.

By Division Lemma 3.1, there exist $N, R \in K[D]$ such that

$$L = M \circ N + R \ , \tag{3.5}$$

where $\mathrm{Sym}(N) = S_2$ and $\mathrm{Sym}(R)$ is either zero or is not divisible by $\mathrm{Sym}(M)$.

We now claim that $R = 0$. Combining (3.3), (3.4), (3.5), we have

$$(M \circ N + R) \circ F + R_1 = M \circ G + R_2 \ ,$$
$$R \circ F + R_1 - R_2 = M \circ (G - N \circ F) \ .$$

Since the orders of R_1, R_2 are both less than the order of F, if R were not zero, the symbol on the left side of the last equation would be that of $R \circ F$, which would imply that $\mathrm{Sym}(R)$ is divisible by $\mathrm{Sym}(M)$ because $\gcd(S_1, S_3) = 1$. Hence $R = 0$, showing that $A = M \circ N \circ F + R_1$ is an incomplete factorization of type $(S_1)(S_2)(S_3)$ with remainder R_1. We now show R_1 is a common obstacle for that type. This follows easily since if $A = M_0 \circ N_0 \circ F_0 + R_0$ is any incomplete factorization of that type, then $A = (M_0 \circ N_0) \circ F_0 + R_0$ is one of type $(S_1 S_2)(S_3)$ and hence $\mathrm{ord}(R_0) \geq \mathrm{ord}(R_1)$. This completes the proof that $V \subseteq W$, which is thus non-empty also. If furthermore, $R_0 \in W$, then since we have shown that $R_1 \in W$ for any $R_1 \in V$, it follows that $\mathrm{ord}(R_0) \leq \mathrm{ord}(R_1)$ and hence R_0 is also of minimal order as a remainder of type $(S_1 S_2)(S_3)$, or in other words, $R_0 \in V$. This shows that $V = W$. $\square$

Example. In Theorem 3.2, take A to be A_4 from (3.2), and take $S_1 = X$, $S_2 = X$, and $S_3 = Y^2$. As we showed in the examples before Theorem 3.2, the orders of common obstacles of the types $(S_1 S_2)(S_3)$ and $(S_1)(S_2 S_3)$ are 1, which is less

then the order of S_3. The theorem implied that the sets of common obstacles to factorization of the types $(S_1 S_2)(S_3)$ and $(S_1)(S_2)(S_3)$ are the same, which accords with our computations in the examples before Theorem 3.2.

Example (Assumptions on the orders of common obstacles are necessary). Consider operator A_4 from (3.2), where $\mathrm{Sym}(A) = X^2Y^2$. Let $S_1 = X^2$, $S_2 = S_3 = Y$. Then $\gcd(S_1, S_3) = 1$. It was shown (example before Theorem 3.2) that with respect to the type $(S_1)(S_2 S_3) = (X^2)(Y^2)$, the operator $R_2 = D_x + D_y + 1$ is the unique common obstacle. Using similar methods, it can be shown that with respect to the type $(S_1 S_2)(S_3) = (X^2 Y)(Y)$, the operator $R_1 = D_x + 1$ is a common obstacle. Here the hypothesis of Theorem 3.2 is not satisfied, because $\mathrm{ord}(R_1) = \mathrm{ord}(R_2) = \mathrm{ord}(S_3) = 1$. It can also be shown that with respect to the type $(S_1)(S_2)(S_3) = (X^2)(Y)(Y)$, the only common obstacle is $R_0 = D_x + D_y + 1$. Clearly, $R_1 \neq R_0$ cannot be a common obstacle of type $(S_1)(S_2)(S_3)$. We note that $N = D_y$, $R = 1$ in this example.

Corollary 3.3. *Let, in $K[D]$, an LPDO A have two factorizations into two factors:*

$$L \circ F = A = M \circ G \quad (\textit{or } F \circ L = A = G \circ M) \ ,$$

where $\gcd(\mathrm{Sym}(F), \mathrm{Sym}(M)) = 1$. *Then there is a factorization of A into three factors:*

$$A = M \circ N \circ F \quad (\textit{or } A = F \circ N \circ M)$$

for some $N \in K[D]$.

Proof. The first statement is implied from that of Theorem 3.2. For the second (the one which is in the brackets) we apply properties of the formal adjoints of LPDOs. □

Example (Fourth Order LPDO). Let

$$\begin{aligned} L &= D_x^3 + (1+x)D_x^2 D_y + x D_x D_y^2 - x^2 D_x^2 - x^3 D_x D_y \\ &\quad + (1-4x)D_x + (x - 2x^2)D_y - 2 \ , \end{aligned}$$

and $F = D_y + x^2$, $M = D_x + x D_y$, and

$$G = D_x^2 D_y + D_x D_y^2 + x^2 D_{xx} + (4x - x^4)D_x + D_y - 4x^3 + x^2 + 2 \ .$$

Then $L \circ F = M \circ G$, meaning that we have two different factorizations into two factors for the LPDO $A = L \circ F$. Moreover, one can find an LPDO N such that $L = M \circ N$. Explicitly, $N = D_x^2 + D_x D_y - x^2 D_x - 2x + 1$. Then $A = M \circ N \circ F$, meaning that A has a factorization into three factors.

Example (Multidimensional LPDO). We have $L \circ F = M \circ G$ for $L = D_x D_y + s D_x + ts + s_x$, $F = D_z + b$, $M = D_x + t$, $G = D_x D_z + b D_y + s D_z + sb + b_y$. It is also easy to see that $L = M \circ N$, where $N = D_y + s$.

Example (Condition $\gcd(\mathrm{Sym}(F), \mathrm{Sym}(M)) = 1$ is necessary for Theorem 3.3). Consider $L = D_x D_y + \frac{1}{1-x} D_x + x D_y + \frac{2-x}{(x-1)^2}$, $F = D_x + \frac{x}{x-1}$, $M = D_x + 1$, $G = D_x D_y + \frac{1}{1-x} D_x + \frac{x^2-x+1}{x-1} D_y - \frac{x}{(x-1)^2}$, for which $L \circ F = M \circ G$. Here $\mathrm{Sym}(L)$ is divisible by $\mathrm{Sym}(M)$, but condition $\gcd(\mathrm{Sym}(F), \mathrm{Sym}(M)) = 1$ fails. On the other hand, the Laplace invariants for LPDO L are $h = -1$, $k = -\frac{-2x+2+x^2}{(x-1)^2} \neq 0$, and, therefore, L has no factorization.

4. Conclusions

The main result of the paper formulated in Theorem 3.2 provides a simplification of the overall picture of factorization of LPDOs.

5. Appendix

Below is an example of how the direct approach and the approach based on Theorem 3.2 are different when it comes to computations.

Let us search for factorizations of the type $(X)(XY)(Y)$ for a bivariate fourth-order LPDO, $A = D_x^2 D_y^2 + \sum_{i+j=0}^{3} a_{ij} D_x^i D_y^j$, $a_{ij} \in K$. The direct approach considers $A = M \circ N \circ F$ for some $M = D_x + m$, $N = D_x D_y + n_{10} D_x + n_{01} D_y + n_{00}$, $F = D_y + f$, where $m, n_{10}, n_{01}, n_{00}, f \in K$. Equating the corresponding coefficients, we have $a_{30} = a_{03} = 0$, $n_{10} = a_{21} - f$, $n_{01} = a_{12} - m$, $n_{00} = mf - ma_{21} - fa_{12} - f_x - (a_{21})_x + a_{11}$, and

$$\left.\begin{array}{rcl} 0 &=& 2f_{xy} - f^2 a_{12} - 4 f_x f + f a_{11} + 2 f_x a_{21} + f_y a_{12} - a_{10} \ , \\ 0 &=& f_y - f^2 + f a_{21} - a_{20} \ , \\ 0 &=& m a_{11} - m^2 a_{21} - 2 m a_{21x} - m_x a_{21} - a_{21xx} + a_{11x} - a_{01} \ , \\ 0 &=& m a_{12} - m^2 - m_x + a_{12x} - a_{02} \ . \end{array}\right\} \tag{5.1}$$

$$\begin{array}{rcl} 0 &=& f_{xxy} - a_{00} + m f_y a_{12} - m^2 f_y - 2 f_x^2 - 2 f_{xx} f + f_{xx} a_{21} + \\ && + f_y a_{12x} + f_{xy} a_{12} - m^2 f a_{21} + m^2 f^2 + f^2 m_x - f^2 a_{12x} - \\ && - f a_{21xx} + f a_{11x} + f_x a_{11} - m f^2 a_{12} - 2 m f a_{21x} + m f a_{11} - \\ && - f m_x a_{21} - 2 f f_x a_{12} - f_y m_x \ , \end{array} \tag{5.2}$$

An approach based on Theorem 3.3 considers $A = L \circ F = M \circ G$ for some $L = D_x^2 D_y + \sum_{i+j=0}^{2} l_{ij} D_x^i D_y^j$, $F = D_y + f$, $M = D_x + m$, $G = D_x D_y^2 + \sum_{i+j=0}^{2} g_{ij} D_x^i D_y^j$, where $l_{ij}, f, m, g_{ij} \in K$. $A = L \circ F$ implies $a_{30} = 0$ and $l_{20} = a_{21} - f$, $l_{02} = a_{03}$, $l_{11} = a_{12}$, $l_{10} = a_{11} - a_{12} f - 2 f_x$, $l_{01} = a_{02} - a_{03} f$, $l_{00} = a_{03} f^2 - f a_{02} - a_{12} f_x - 2a - 03 f_y - f_{xx} + a_{01}$, while $A = M \circ G$ implies $a_{03} = 0$, and $g_{20} = 0$, $g_{11} = a_{21}$, $g_{02} = a_{12} - m$, $g_{10} = a_{20}$, $g_{01} = a_{11} - m a_{21} - a_{21x}$, $g_{00} = a_{10} - m a_{20} - a_{20x}$. The remaining conditions are

conditions (5.1) ,

and two new conditions:

$$0 = a_{00} - ma_{10} + m^2 a_{20} + 2ma_{20x} - a_{10x} + m_x a_{20} + a_{20xx} \ , \qquad (5.3)$$

$$\begin{aligned} 0 = \; & f_{xxy} - f^2 a_{02} - 2f_x a_{12} f - 2f_{xx} f + f a_{01} - 2f_x^2 + f_x a_{11} + \\ & + f_y a_{02} + f_{xx} a_{21} + a_{12} f_{xy} - a_{00} \ . \end{aligned} \qquad (5.4)$$

Thus, when algebraic manipulations only are used the difference between the two approaches applied to the given problem is as follows. Instead of the non-linear Partial Differential Equation (PDE) in two unknown variables f and m, (5.2) that we have in the first (direct) approach, the second approach implies a non-linear PDE in variable f, (5.4) and another one in variable m, (5.3). In other words, the second approach gives separation of variables.

References

[1] I. Anderson and M. Juras. Generalized Laplace invariants and the method of Darboux. *Duke J. Math.*, 89:351–375, 1997.

[2] I. Anderson and N. Kamran. The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane. *Duke J. Math.*, 87:265–319, 1997.

[3] C. Athorne. A $Z^2 \times R^3$ Toda system. *Phys. Lett. A.*, 206:162,–166, 1995.

[4] S.Y. Startsev A.V. Zhiber. Integrals, solutions and existence of the Laplace transformations for a linear hyperbolic system of equations. *Math. Notes*, 74(6):848–857, 2003.

[5] H. Blumberg. *Über algebraische Eigenschaften von linearen homogenen Differentialausdrücken*. PhD thesis, Göttingen, 1912.

[6] T. Cluzeau. Factorization of differential systems in characteristic p. In *ISSAC '03: Proceedings of the 2003 international symposium on Symbolic and algebraic computation*, pages 58–65, New York, NY, USA, 2003. ACM.

[7] T. Cluzeau and A. Quadrat. Using morphism computations for factoring and decomposing general linear functional systems. In *Proceedings of MTNS 2006, Kyoto (Japan)*, 2006.

[8] T. Cluzeau and A. Quadrat. Factoring and decomposing a class of linear functional systems. *Linear Algebra and its Applications*, 428(1):324 –381, 2008.

[9] G. Darboux. *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, volume 2. Gauthier-Villars, 1889.

[10] E. Shemyakova. Multiple factorizations of bivariate linear partial differential operators. *LNCS*, 5743:299–309, 2009.

[11] D. Grigoriev and F. Schwarz. Factoring and solving linear partial differential equations. *Computing*, 73(2):179–197, 2004.

[12] D. Grigoriev and F. Schwarz. Generalized Loewy-decomposition of D-modules. In *ISSAC '05: Proceedings of the 2005 international symposium on Symbolic and algebraic computation*, pages 163–170, New York, NY, USA, 2005. ACM.

[13] D. Grigoriev and F. Schwarz. Loewy decomposition of third-order linear pde's in the plane. In *ISSAC '08: Proceedings of the 2005 international symposium on Symbolic and algebraic computation*, pages 277–286, New York, NY, USA, 2008. ACM.

[14] Z. Li, F. Schwarz, and S.P. Tsarev. Factoring zero-dimensional ideals of linear partial differential operators. In *ISSAC '02: Proceedings of the 2002 international symposium on Symbolic and algebraic computation*, pages 168–175, New York, NY, USA, 2002. ACM Press.

[15] Ziming Li, Fritz Schwarz, and Serguei P. Tsarev. Factoring systems of linear PDEs with finite-dimensional solution spaces. *J. Symb. Comput.*, 36(3-4):443–471, 2003.

[16] P.J. Cassidy and M.F. Singer. A Jordan-Hölder theorem for differential algebraic groups. *submitted to the Journal of Algebra*, 1. arXiv:1003.3274, April 2010.

[17] E. Shemyakova. The parametric factorizations of second-, third- and fourth-order linear partial differential operators on the plane. *Mathematics in Computer Science*, 1(2):225–237, 2007.

[18] E. Shemyakova and F. Winkler. Obstacle to factorization of LPDOs. In J.-G. Dumas, editor, *Proc. Transgressive Computing 2006, Conference in Granada Spain*, pages 435–441, Grenoble, France, 2006. Universite J. Fourier.

[19] E. Shemyakova and F. Winkler. Obstacles to the factorization of linear partial differential operators into several factors. *Programming and Computer Software*, 33(2):67–73, 2007.

[20] S.Y. Startsev. Cascade method of Laplace integration for linear hyperbolic systems of equations. *Mathematical Notes*, 83, 2008.

[21] S. Tsarev. Factorization of linear partial differential operators and Darboux' method for integrating nonlinear partial differential equations. *Theo. Math. Phys.*, 122:121–133, 2000.

[22] S. P. Tsarev. An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator. In *ISSAC '96: Proceedings of the 1996 international symposium on Symbolic and algebraic computation*, pages 226–231, New York, NY, USA, 1996. ACM.

[23] S.P. Tsarev. Generalized Laplace transformations and integration of hyperbolic systems of linear partial differential equations. In *ISSAC '05: Proceedings of the 2005 international symposium on Symbolic and algebraic computation*, pages 325–331, New York, NY, USA, 2005. ACM Press.

[24] S.P. Tsarev and E. Shemyakova. Differential transformations of parabolic second-order operators in the plane. *Proceedings Steklov Inst. Math. (Moscow)*, 266:219–227, 2009. see also electronically http://arxiv.org/abs/0811.1492.

Acknowledgments. The author was supported by the Austrian Science Fund (FWF) under project DIFFOP, Nr. P20336-N18.

Ekaterina Shemyakova
Research Institute for Symbolic Computation (RISC)
J.Kepler University
Altenbergerstr. 69
A-4040 Linz
Austria
e-mail: `Ekaterina.Shemyakova@risc.jku.at`